A countable union friendly version of Minkowski dimension: packing dimension

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Let h be a gauge function. Define h-packing pre-measure

$\tilde{P}^h(K) := \lim_{\varepsilon \to 0} \left(\sup \sum_{j=1} h(2r_j) \right)$, where sup is taken over collections of disjoint $(B(x_j, r_j))_{j=1}^{\infty}$ with $x_j \in K$, $r_j < \varepsilon$.

$\tilde{P}^h$ is finitely sub-additive, but not countably!

h-Packing measure is defined as

$P^h(K) := \inf \{ \sum_{j=1}^{\infty} \tilde{P}^h(K_j),\ K \subset \bigcup_{j=1}^{\infty} K_j \}$. Remark: K-countable $\Rightarrow P^h(K) = 0$.

P^h is a metric outer measure, so all Borel are measurable.

$P^\alpha(K) := P^{t^\alpha}(K)$ α-dim packing measure.

As usual, define Packing dimension of K as

$\mathrm{Pdim}\, K = \inf \{ \alpha : P^\alpha(E) = 0 \}$.

The main property.

$\mathrm{Pdim}\, A = \inf \{ \sup_{j \ge 1} \overline{\mathrm{Mdim}}(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j \}$.

Corollary. $\mathrm{Hdim}\, A \le \mathrm{Pdim}\, A \le \overline{\mathrm{Mdim}}\, A$.

Pf (of Thm). Take $E \subset A$

($\ge$) Assume $\tilde{P}^\alpha(E) < \infty$. Then $\exists P(\varepsilon, E)$ non-intersecting balls of radius ε. So

$\sup P(\varepsilon, E)(2\varepsilon)^\alpha \le \tilde{P}^\alpha(E) < \infty$. So $\overline{\mathrm{Mdim}}(E) \le \alpha$.

If $\mathrm{Pdim}\, A < \alpha$, then $A = \bigcup A_j$, with $\tilde{P}^\alpha(A_j) < \infty$ (it is first $A \subset \bigcup E_j$, $P^\alpha(E_j) = 0$; and then each E_j is a countable union of sets with finite $P^\alpha(A_j)$).

Then $\overline{\mathrm{Mdim}}\, A_j \le \alpha$, implying ($\ge$).

($\le$) Assume $\tilde{P}^\alpha(E) > 0$. Let us prove that $\overline{\mathrm{Mdim}}\, E \ge \alpha$.

$P^\alpha(E) > 0 \Rightarrow \overline{\mathrm{Mdim}}\, E \ge \alpha$

Indeed, for some $s > 0$ and $\forall \varepsilon > 0$, we can find a collection $(B(x_j, r_j))$, $x_j \in E$, $r_j < \varepsilon$, $\sum r_j^\alpha > s$.

Let $N_m = \#\{ j : 2^{-m-1} \le r_j < 2^{-m} \}$ (we fix ε and covering, for $m < m_0 = \lfloor \log_2 \frac{1}{\varepsilon} \rfloor$, $N_m = 0$).

Then

1) $N_m \le P(2^{-m-2}, E)$

2) $\sum N_m 2^{(-m-1)\alpha} \ge \sum \left(\frac{r_j}{2}\right)^\alpha > 2^{-\alpha} s$.

So we get

$\sum_{m \ge m_0} P(2^{-m-2}, E)\, 2^{(-m-2)\alpha} > s_1 := 4^{-\alpha} s$, so $\sum P(2^{-m}, E) 2^{-m\alpha} = \infty$!

Thus, if $\beta < \alpha$, then $\overline{\lim} \frac{\log P(2^{-m-2}, E)}{(m+2)\log 2} > \beta$ (otherwise, the $P(2^{-m-2}, E)\, 2^{(-m-2)\alpha}$ would decay exponentially). So $\overline{\mathrm{Mdim}}\, E \ge \alpha$.

Now if for some cover $E = \bigcup E_j$, we have

$\sup \overline{\mathrm{Mdim}}\, E_j < \alpha \Rightarrow \tilde{P}^\alpha(E_j) = 0\ \forall j \Rightarrow P^\alpha(E) = 0 \Rightarrow \mathrm{Pdim}\, E \le \alpha$ ∎

Observe a very useful thing, distinguishing Pdim from Mdim:

$\mathrm{Pdim} \bigcup_{j=1}^{\infty} A_j = \sup \mathrm{Pdim}\, A_j$,

But also $\tilde{P}^h(A) = \tilde{P}^h(\bar{A})$, so we can always consider coverings by closed sets to compute P^h and Pdim!

An interesting example:

$A_S := \{ \sum_{j \in S} x_j 2^{-j};\ x_j \in \{0,1\} \}$, as before, for $S \subset \mathbb{N}$.

Then $\mathrm{Pdim}\, A_S = \overline{\mathrm{Mdim}}\, A_S = \bar{d}(S)$

Proof.

Let us consider any cover $A_S \subseteq \bigcup A_j$ by closed sets.

By Baire category theorem, $\exists A_j$: relative interior of A_j is not empty, i.e. for some $x \in A_S$, $m \in \mathbb{N}$, $B(x, 2^{-m}) \cap A_S \subset A_j$.

But if $y = \sum_{j \in S} y_j 2^{-j}$, with $y_j = x_j$, $j \le m$; $y_j \in \{0,1\}$, $j > m$, then $y \in B(x, 2^{-m})$.

Thus if $T = S \cap \{n > m\}$, then $\tilde{x} + A_T \subset B(x, 2^{-m})$, where $\tilde{x} = \sum_{j \in S, j \le m} x_j 2^{-j}$.

Thus $\overline{\mathrm{Mdim}}\, A_j \ge \overline{\mathrm{Mdim}}\, A_T = \bar{d}(T) = \bar{d}(S)$

So we can have $\mathrm{Pdim}\, A_S > \underline{\mathrm{Mdim}}\, A_S$.